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NON-LINEAR HEAT GENERATION AND STABILITY OF THE TEMPERATURE DISTRIBUTION IN CONDUCTING SOLIDS

DANIEL D. JOSEPH

Department of Aeronautics and Engineering Mechanics, University of Minnesota, Minneapolis, Minn., U.S.A.

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Abstract—The effect of non-linear dependence of resistance on temperature on the Joulean production of heat in electrically conducting systems is investigated. The theory is compared with well-known linear theories. In common conducting materials there exists a critical current beyond which steady solutions are unavailable. Unlike the linear theory the critical current does not imply unbounded temperatures. The linear theory always overestimates the critical current. In the non-linear theory the solutions for currents below critical are not unique. The various branches of the non-unique solutions are not all stable. A neutral solution and neighboring unstable solutions to the associated stability problem exist when the current is at the critical value.

NOMENCLATURE

$2h$,	channel separation distance;
I,	current;
k,	thermal conductivity;
R_0,	resistance at temperature of the wall;
V,	volume;
x, y, z,	$(x_1/h,\ x_2/h,\ x_3/h)$ dimensionless co-ordinates;
t,	dimensionless time;
ψ,	dimensionless temperature difference referred to wall temperature;
ψ_m,	maximum of ψ;
$\dot{\psi}$,	$\partial\psi/\partial\psi_m$;
ψ',	perturbation temperature difference;
θ,	temperature difference referred to wall temperature;
$\bar{\theta}$,	$(\mathrm{d}\phi/\mathrm{d}\theta)^{-1}_{\theta=0}$;
$\phi(\psi)$,	dimensionless heat source (resistance);
$\phi'(\psi)$,	$\mathrm{d}\phi/\mathrm{d}\psi$;
$\phi''(\psi)$,	$\mathrm{d}^2\phi/\mathrm{d}\psi^2$;
λ,	$I^2R_0h/k\bar{\theta}V$;
$\dot{\lambda}$,	$\mathrm{d}\lambda/\mathrm{d}\psi_m$;

INTRODUCTION

A WELL known result of the theory of steady heat conduction in electrical coils generating Joulean heat is that if the resistance is assumed linear in the temperature and the current increased slowly to a certain finite value, the coil temperature grows without bound. Though the mathematics in this problem resembles that which described the buckling of Euler columns, it seems not to have been recognized that a stability problem is also involved. Moreover, the effects of a general dependence of resistance on temperature have not been analyzed. We show in this paper how the salient features of the critical current phenomena are controlled by the assumed nature of the dependence of resistance on temperature. The existence of the critical current is interpreted from the point of view of the stability of the steady temperature distribution.

In Section 2 of this paper we treat non-linear generation of heat in a plane electrically conducting plate. When the resistance is proportional to the first or a greater power of the temperature and the wall temperatures prescribed, there exists a finite critical value of the current beyond which steady solutions do not exist. This feature is clearly apparent in each of the substantial number of exact solutions which have been developed for the linear case [1, 2]. For the non-linear case there is the additional complication that the solutions which do exist are not unqiue. To each current below the critical belong two or more solutions characterized by a different maximum temperature. As in the linear case no solutions exist when the current exceeds a certain finite critical value. Unlike the linear case this critical current corresponds to a finite maximum temperature. If the resistance function also has an increasing curvature (as for common metals) there exists a first maximum of the current in the neighborhood of which the solution is double valued. Through this maximum (Section 3) there is a neutral (zero wave number) solution to the associated stability equation. Neighboring the neutral solution are stable (unstable) solutions which for a fixed current below the critical are associated with a lower (higher) maximum temperature.

In Section 4 of the paper the results which apply to the plane plate are extended to other geometries and made to include certain effects of non-homogeneity. It is shown that the critical current associated with the linear resistance universally bounds the critical current associated with the non-linear problem.

An exact solution corresponding to a quadratic polynomial dependence of resistance on temperature is constructed in the appendix.

The results of this investigation show that the critical current phenomenon is not a mere consequence of the linearized resistance and is not necessarily associated with infinite (or even large) temperature differences. An alternative view of the critical current is discussed in the conclusion. Similar results hold for fluid systems which generate heat by viscous friction [4, 5, 6] and will certainly apply to certain exothermic chemically reacting systems.

2. PLANE PLATE

We shall develop the theory in detail for the simple case of an infinitely wide plane plate. The use of this geometry as a model for an electrical coil is discussed by Jakob [1]. In Section 4 we extend the results of this section to other geometries.

The co-ordinate origin is located at the plate center. The top and bottom of the plate are held at a common temperature. The current flow is parallel to walls and the heat generation which is assumed equal to electrical power dissipation is represented by the non-dimensional temperature dependent function $\phi(\psi)$. It is assumed that the thermal conductivity is constant. In dimensionless variables we have

$$\frac{d^2\psi}{dx^2} = -\lambda\,\phi(\psi) \tag{1}$$

$$\psi(1) = \psi(-1) = 0 \tag{2}$$

where $\lambda = I^2R_0h^2/k\bar{\theta}V$ is the current parameter. The reference temperature $\bar{\theta}$ is chosen so that polynomial approximations to ϕ may be written as $\phi_m = 1 + \psi + \delta_2\psi^2 + \cdots \delta_m\psi^m$. We assume that the resistance is a monotonically increasing function of the temperature so that $\phi'(\psi) \geqslant 1$ with $\phi'(0) = 1$.

Since $\phi(\psi) \geqslant 1$ the curvature of ψ is negative. There is but one stationary point in $(-1, 1)$ and it is a maximum. By symmetry the maximum occurs at the channel center, and we may replace the conditions (2) with

$$\frac{d\psi}{dx}(0) = \psi(1) = 0. \tag{3}$$

The symmetry requirement, though convenient, is not essential in what follows. Suppose at the upper wall $\psi(1) \neq 0$. Then $\Gamma = \psi - \psi(1)(1 + x)/2$ satisfies

$$\frac{d^2\Gamma}{dx^2} + \lambda\,\phi\,[\Gamma + \psi(1)(1 + x)/2]$$

$$\Gamma(1) = \Gamma(-1) = 0$$

and from the negative curvature and boundary conditions there exists $x = \xi(-1 \leqslant \xi \leqslant 1)$ such that

$$\frac{d\Gamma}{dx}(\xi) = 0.$$

A linear transformation of co-ordinates and a redefinition of λ could then be used to restore the conditions (3) for Γ.

Equation (1) and conditions (3) are placed by the equivalent integral equation

$$\psi = \lambda \int_x^1 d\eta \int_0^\eta \phi\,[\psi(\gamma)]\,d\gamma. \tag{4}$$

Since ψ is convex

$$\psi_m(1 - x) \leqslant \psi \leqslant \psi_m$$

where $\psi_m = \psi(0)$. This last inequality is combined with (4) to produce

$$\lambda \int_x^1 d\eta \int_0^\eta \phi\,[\psi(1 - \gamma)]\,d\gamma \leqslant \psi \leqslant \lambda \int_x^1 d\eta \int_0^\eta \phi\,[\psi_m]\,d\gamma \tag{5}$$

with λ restricted by the condition that (5) holds when $x = 0$

$$LB(\lambda) = \frac{2\,\psi_m}{\phi\,(\psi_m)} \leqslant \lambda \leqslant \frac{\psi_m}{\int\limits_0^1 \mathrm{d}\eta \int\limits_0^\eta \phi\,[\psi_m\,(1-\gamma)]\,\mathrm{d}\gamma} = UB\,(\lambda). \quad (6)$$

It follows from (6) that no solutions exist when the current parameter exceeds max $[UB(\lambda)]$. The behavior of the solutions depends largely on the order with which $\phi(\psi_m)$ increases as ψ_m tends to infinity. We let ϕ_{k_v} represent the asymptotic development of ϕ so that

$$\lim_{\psi_m \to \infty} \phi\;[\psi_m] \to A_\nu\,\psi_m^{k_\nu}\;(A_v > 0)$$

and from (6)

$$\lim_{\psi_m \to \infty} \left\{\frac{2\,\psi_m}{A_\nu\,\psi_m^{k_\nu}} \leqslant \lambda \leqslant \frac{k_v + 2}{A_\nu\,\psi_m^{k_\nu}}\,\psi_m\right\}.$$

Three cases may be distinguished.

(a) $0 \leqslant k_\nu \leqslant 1$, $\lambda \to \infty$. The current is a unique and increasing function of the maximum temperature. Solutions exist for all λ.

(b) $k_\nu = 1$, $2/A_\nu \leqslant \lambda \leqslant 3/A_\nu$. The current is a unique and increasing function of the maximum temperature but possesses a finite asymptote $(3/A_v)$ beyond which solutions do not exist.

(c) $k_\nu > 1$, $\lambda \to 0$. The current is not a unique function of the maximum temperature. $UB(\lambda)$ has two zeros and must possess at least one maximum. Beyond max $LB(\lambda)$ there are no solutions. Below max $LB(\lambda)$ there are at least two solutions.

For our purposes it will suffice to remark that there are many materials for which the conclusions of (c) above apply. For many of these $\phi(\psi) \geqslant 1, \phi'(\psi) \geqslant 1, \phi''(\psi) \geqslant 0$ where the equality applies only for $\psi = 0$. Iron, tungsten and gold are but a few of the metals which satisfy the above conditions over a wide range of temperatures.

That a solution of equation (1) and the boundary conditions exists for all ψ_m may be easily established by direct quadrature. Thus we have

$$\lambda^{1/2} = \int\limits_0^{\psi_m} \frac{\mathrm{d}\psi}{\{G\,(\psi_m) - G(\psi)\}^{1/2}},$$

where $G = \int \phi(\psi)\,\mathrm{d}\psi$,

so that given any ψ_m we may calculate a corresponding λ.

We now establish that the first stationary point of $\lambda\,(\psi_m)$ is maximum when $\phi'' \geqslant 0$. From (1) and (3) $(\dot\psi = \partial\psi/\partial\psi_m)$

$$\frac{\mathrm{d}^2\dot\psi}{\mathrm{d}x^2} = -\,\dot\lambda\,\phi - \lambda\,\phi'\,\dot\psi \quad (7)$$

$$\frac{\mathrm{d}^2\ddot\psi}{\mathrm{d}x^2} = -\,\ddot\lambda\,\phi - 2\,\dot\lambda\,\phi'\dot\psi - \lambda\,\phi''\,\dot\psi^2 - \lambda\,\phi'\ddot\psi \quad (8)$$

$$\dot\psi\,(1) = \frac{\mathrm{d}\dot\psi}{\mathrm{d}x}(0) = \ddot\psi\,(1) = \frac{\mathrm{d}\ddot\psi}{\mathrm{d}x}(0) = 0$$

Multiply (7) by $\ddot\psi$ and (8) by $\dot\psi$ and integrate over (0, 1) to obtain $(\dot\lambda = 0)$

$$\ddot\lambda_{\dot\lambda=0} = -\,\lambda\,\frac{\int\limits_0^1 \phi''\,\dot\psi^3\,\mathrm{d}\gamma}{\int\limits_0^1 \phi\,\dot\psi\,\mathrm{d}\gamma} \quad (9)$$

We observe that with $\dot\lambda \to 0$ and $\dot\psi\,(0) = 1$ (7) must generate $\dot\psi \geqslant 0$, $(\mathrm{d}\dot\psi/\mathrm{d}x) \leqslant 0$. Moreover (7) cannot generate $\dot\psi(x)$ with an interior zero unless $\mathrm{d}\dot\psi/\mathrm{d}x$ also possesses an interior zero. But

$$\frac{\mathrm{d}\dot\psi}{\mathrm{d}x} = -\,\dot\lambda \int\limits_0^x \phi\,(\psi)\,\mathrm{d}x - \lambda \int\limits_0^x \phi'\,\dot\psi\,\mathrm{d}x$$

cannot vanish before $\dot\lambda$ changes sign. This implies that $\dot\psi \geqslant 0$ when $\dot\lambda$ first vanishes. That this point is maximum follows from $\ddot\lambda < 0$. The possibility that $\ddot\lambda > 0$ at other stationary points would also seem very restricted and only this one maximum appears in the exact solutions (see appendix).

The behavior of solutions and of the function $\lambda\,(\psi_m)$ can be quite closely approximated by the bounding technique which leads to (6). The bounds can be replaced with a tighter relation. From (4) we form the ratio

$$\frac{\psi_m}{\psi} = \frac{\int\limits_0^1 \mathrm{d}\eta \int\limits_0^\eta \phi\,(\psi)\,\mathrm{d}\gamma}{\int\limits_x^1 \mathrm{d}\eta \int\limits_0^\eta \phi\,(\psi)\,\mathrm{d}\gamma} = 1 + \frac{\int\limits_0^x \eta\,\mathrm{d}\eta\,\phi\,(\eta)}{\int\limits_x^1 \eta\,\mathrm{d}\eta\,\phi\,(\eta)}$$

$$\geqslant 1 + \frac{\int\limits_0^x \eta \, d\eta \min \phi(\eta)}{\int\limits_1^x \eta \, d\eta \max \phi(\eta)} = \frac{1}{1 - x^2}.$$

Hence

$$\psi_m (1 - x) \leqslant \psi(x) \leqslant \psi_m (1 - x^2)$$

and the solution is found in a region bounded by parabolas of the first and second degree. It then follows from (4) that λ must lie in a region defined by

$$\frac{\psi_m}{\int\limits_0^1 d\eta \int\limits_0^\eta \phi[\psi_m(1-\gamma^2)]\, d\gamma} \leqslant \lambda \leqslant \frac{\psi_m}{\int\limits_0^1 d\eta \int\limits_0^\eta \phi[\psi_m(1-\gamma)]\, d\gamma}. \qquad (10)$$

In Fig. 1 we have used (10) to bound the exact solution for ϕ_2 (see appendix). The bounds (10) are evaluated as

$$\frac{2\psi_m}{1 + \frac{5}{6}\psi_m + \frac{11\delta}{15}\psi_m^2} \leqslant \lambda \leqslant \frac{2\psi_m}{1 + \frac{2}{3}\psi_m + \frac{\delta\psi_m^2}{2}}. \qquad (11)$$

We have also plotted the exact solution corresponding to ϕ_1 and ϕ_0 giving a graphical representation of the conclusions of this section.

3. STABILITY OF THE TEMPERATURE DISTRIBUTION

The equation governing the unsteady conduction of heat generated by an arbitrary source (resistance) function is

$$\frac{\partial \psi^*}{\partial t} = \nabla^2 \psi^* + \lambda \phi(\psi^*). \qquad (12)$$

The perturbation $\psi'(x, y, z, t)$ from the steady-state solution $\psi(x)$ is introduced, i.e. $\psi^* = \psi' + \psi$. We drop quadratic powers of ψ' and expand ϕ to obtain

$$\frac{\partial \psi'}{\partial t} = \nabla^2 \psi' + \lambda \phi'(\psi)\psi'$$
$$\phi'(\psi) = d\phi/d\psi \qquad (13)$$

and require that

$$\psi'(-1, y, z, t) = \psi'(1, y, z, t) = 0. \qquad (14)$$

We next seek solutions for wavy disturbances

$$\psi' = \psi_c(x) \exp[-ct + iay + i\beta z] \qquad (15)$$

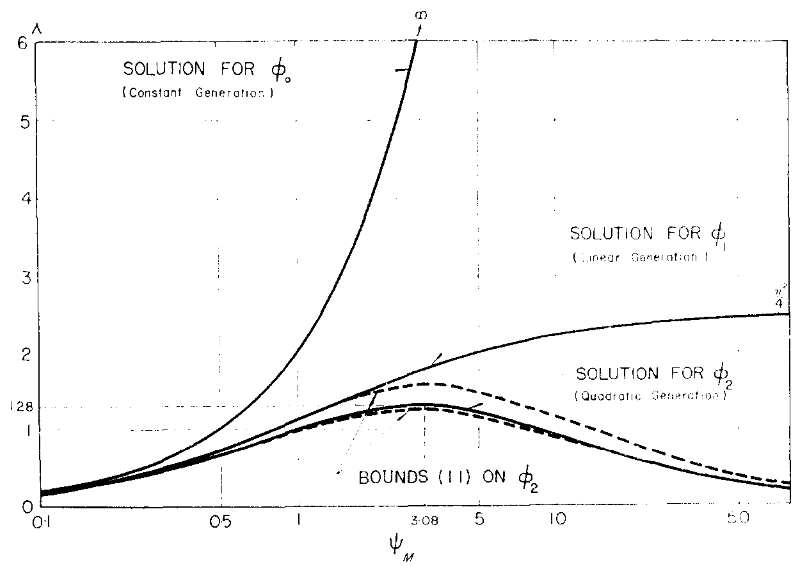

FIG. 1. Current parameter as function of maximum temperature.

and infer that

$$\frac{d^2\psi_c}{dx^2} = \{\alpha^2 + \beta^2 - c - \lambda\phi'(\psi)\}\,\psi_c \quad (16)$$

$$\psi_c(1) = \psi_c(-1). \quad (17)$$

Equations (16) and (17) constitute an eigenvalue problem from which we may determine stable ($c > 0$) and unstable ($c < 0$) eigenvalues.

Consider first the linear case $\phi'(\psi) = 1$. A complete set eigenfunctions which satisfy (17) are

$$\cos \nu\pi x/2 \; (\nu = 1, 3, 5 ----)$$

$$\sin \nu\pi x/2 \; (\nu = 2, 4, 6 ----)$$

and these satisfy (16) if

$$c + \lambda - \alpha^2 - \beta^2 = \left(\frac{\nu\pi}{2}\right)^2.$$

The least eigenvalue is

$$c = \pi^2/4 + \alpha^2 + \beta^2 - \lambda(\psi_m).$$

Several conclusions may be drawn:

1. The temperature distribution is stable with $\lambda(\psi_m) < \pi^2/4$. From considerations of the preceding section we know that this condition must always hold.
2. There is a neutral solution for zero wave numbers and

$$\lambda(\psi_m \to \infty) \to \pi^2/4.$$

It also follows that the temperature distribution is less stable to layered (zero wave number) disturbances than to periodic disturbances.

Now we consider the non-linear case

$$[\phi'(\psi) \geqslant 1,\; \phi''(\psi) \geqslant 0]$$

for the condition of greatest instability

$$(\alpha = \beta = 0).$$

We first observe that *there is a neutral solution of* (16) *and* (17) *where* $d\lambda/d\psi_m = 0$. This follows from comparison of equations (7) and (8)

$$\frac{d^2\psi}{dx^2} = -\lambda\phi - \lambda\phi'(\psi)\,\psi$$

$$\psi(1) = \psi(-1) = 0$$

and (16) and (17). When $\lambda = c = 0$ the solution of (16) and (17) is given by

$$A\,\psi_c(x) = \psi(x)$$

where A is an arbitrary constant. A neutral solution of the perturbation equation is thus associated with the critical current. In particular this is true in the neighborhood of the first maximum of $\lambda(\psi_m)$.

Now we examine the behavior of $c[\lambda(\psi_m)]$ in the neighborhood of this first maximum. From equations (8) and (16) we obtain

$$0 = \int_{-1}^{1} \left\{\psi \frac{d^2\psi_c}{dx^2} - \psi_c \frac{d^2\psi}{dx^2}\right\} dx$$

$$= \int_{-1}^{1} \{\lambda\,\phi(\psi)\,\psi_c - c\psi_c\,\psi\}\,dx.$$

This is rewritten as

$$\frac{c}{\lambda} = \frac{\int_{-1}^{1} \phi(\psi)\,\psi_c\,dx}{\int_{-1}^{1} \psi_c\,\psi\,dx}$$

and in the passage to the limit

$$\frac{dc}{d\lambda} = \frac{\int_{-1}^{1} \phi(\psi)\,\psi\,dx}{\int_{-1}^{1} \psi^2\,dx}. \quad (18)$$

It follows that in *the neighborhood of the first maximum of* $\lambda(\psi_m)$ *the eigenvalue c and the slope* λ *have the same sign.*

Existing exact solutions which satisfy the condition $\phi'(\psi) \geqslant 1$, and $\phi''(\psi) \geqslant 0$ have only one such stationary point and there are just two branches of the solution. The first branch is stable and the second unstable.

The second branch has a higher maximum temperature and could presumably be started by preheating and maintained by large dissipation with small currents. The high temperatures are certainly unstable and if disturbed would decrease to values compatible with the stable solution at the given current.

4. OTHER GEOMETRIES—INHOMOGENEOUS MATERIALS

The results of the previous sections are here extended to round wires, spheres and certain inhomogeneous materials. We again consider materials for which $\phi'(\psi) \geqslant 1$, and $\phi''(\psi) \geqslant 0$. For these the generation (resistance) function may be represented by

$$\phi(\psi) = \psi + G(\psi)$$
$$G(\psi) > 1$$
$$G(0) = 1.$$

The steady equation governing the temperature

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{p(x)\frac{\mathrm{d}\psi}{\mathrm{d}x}\right\} + \lambda f(x)\left[\psi + G(\psi)\right] = 0 \tag{19}$$
$$\left.\begin{matrix} p(x) > 0 \\ f(x) > 0 \end{matrix}\right\} a \leqslant x \leqslant b$$

and the boundary conditions

$$\psi(a) = \psi(b) = 0 \tag{20}$$

resemble the Sturm–Liouville system to which they reduce with $G = 0$. For solid spheres or cylinders we replace (20) with

$$\frac{\mathrm{d}\psi}{\mathrm{d}x}(a = 0) = \psi(b = 1) = 0. \tag{21}$$

These latter cases are singular $[p(0) = f(0) = 0]$ but this introduces no essential modification of the results which follow.

Let $\hat{\psi}$ be the solution to the reduced linear system

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{p(x)\frac{\mathrm{d}\hat{\psi}}{\mathrm{d}x}\right\} + \hat{\lambda} f(x)\,\hat{\psi} = 0 \tag{22}$$

and either of the conditions (20) or (21). From the Sturm–Liouville theory it follows that the linear system will generate a sequence of positive eigenvalues. To the first such value $\hat{\lambda}_0$ will correspond an eigenfunction $\hat{\psi}_0$ which satisfies the boundary conditions and has no zero in (a, b).

Assume that a solution of (19) and (20) or (21) exists when $\lambda = \hat{\lambda}_0$. From (19) and (22) we obtain

$$0 = \int_a^b \left\{\hat{\psi}_0 \frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}\psi}{\mathrm{d}x}\right) - \psi\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}\hat{\psi}_0}{\mathrm{d}x}\right) + \hat{\lambda}_0 f(x)\,\hat{\psi}_0\, G(\psi)\right\}\mathrm{d}x$$

and integrating by parts

$$0 = \hat{\lambda}_0 \int_a^b f(x)\,\hat{\psi}_0(x)\, G(\psi)\,\mathrm{d}x > 0.$$

It follows that our assumption was erroneous and that there is no solution of (19) and (20) or (21) when $\lambda = \hat{\lambda}_0$ (the first eigenvalue of the reduced linear system).

Now let $\lambda \neq \hat{\lambda}_0$ be any value of λ for which the nonlinear system has a solution. From (19) and (20)

$$\frac{\lambda}{\hat{\lambda}_0} = \left\{1 + \frac{\int_a^b f\,\hat{\psi}_0\, G\,\mathrm{d}x}{\int_a^b f\psi\,\hat{\psi}_0\,\mathrm{d}x}\right\} \leqslant \left\{1 + \frac{1}{\psi_m}\right\}^{-1} = \frac{\psi_m}{\psi_m + 1}$$

It follows that $\hat{\lambda}_0$ is a universal upper bound on $\lambda(\psi_m)$ and that

$$\lambda(\psi_m) \leqslant \hat{\lambda}_0\, \psi_m/(\psi_m + 1).$$

For circular wires $\hat{\lambda}_0 = 5{\cdot}784$ as determined by the first positive root of $J_0(\lambda^{1/2}) = 0$. For the infinite plate $\hat{\lambda}_0 = \pi^2/4$ (see Fig. 1).

The universal bounds, though independent of the (suitably restricted) functional form of ϕ are not as tight as those developed by the integral technique of Section 2.

Note that the functions $p(x)$ and $f(x)$ of equation (19) include effects of position dependent conductivity and resistivity so that these effects are accommodated in this formulation.

The remarks leading to equation (9) apply in the general case so that

$$\ddot{\lambda}_{\dot{\lambda}=0} = -\lambda \frac{\int_a^b f\phi''\,\psi^3\,\mathrm{d}x}{\int_a^b f\phi\,\psi\,\mathrm{d}x}.$$

There is no interior zero of ψ before the first ψ_m for which $\dot{\lambda} = 0$. This point is a maximum and in its neighborhood the solution a double valued function of λ. The possibility that for larger values of ψ_m the function $\lambda(\psi_m)$ assumes a relative minimum also seems quite restricted.

The conclusions relative to the problem of stability of the temperature distribution also carry over. A neutral solution for layered disturbances coincides with each stationary point of $\lambda(\psi_m)$. The remarks which lead to (18) hold and we obtain

$$\frac{\mathrm{d}c}{\mathrm{d}\lambda}_{\lambda=0} = \frac{\int_a^b f\,\phi\,\psi\,\mathrm{d}x}{\int_a^b g\,\psi^2\,\mathrm{d}x}$$

where $g(x) > 0$ in (a, b) and for isotropic materials

$$g(x) = f(x) = \begin{cases} x & \text{cylindrical co-ordinates} \\ x^2 & \text{spherical co-ordinates.} \end{cases}$$

Solutions immediately to the right of the first maximum of $\lambda(\psi_m)$ are stable. Those to the left are unstable.

5. CONCLUSIONS

A critical current is a current beyond which steady-state solutions to the conduction equation with wall temperatures prescribed do not exist. A critical current will exist for heat generation (resistance) functions proportional to a linear or higher power of the temperature. When the order of this dependency is greater than or equal to zero but less than one (constant to linear generation), no critical current exists. The linear theory always overestimates the value of the critical current. Moreover in the non-linear case the critical current coincides with a finite (not necessarily large) value of the maximum temperature. Hence the unbounded temperatures, which are commonly used to define the critical current, are only accidental and not essential to the phenomenon.

In the non-linear case the temperature distribution which develops is not a unique function of the current. Conceivably distributions with high temperatures could be attained by preheating and maintained with small currents. But such temperature distributions are unstable and if disturbed would presumably assume lower stable values compatible with the given current.

APPENDIX—EXACT SOLUTIONS

Exact solutions of the non-linear problem are sparse. For $\phi = e^{\psi}$ there is a solution for the plane plate which has been examined by Jakob [5]. A simple solution for this resistance function has also been used by Kearsley [7] to discuss variable viscosity effects in the flow of fluids in round pipes and is easily adapted to wires. The plane plate problem is particularly simple because it possesses an energy integral. However, even here solutions in terms of tabulated functions are scarce. It is possible to obtain the solution for this latter problem when $\phi = \phi_2$ or $\phi = \phi_3$, i.e. quadratic or cubic polynomials, in terms of elliptic functions. Below we exhibit the solution of (1) and (3) with $\phi = \phi_2 = 1 + \psi + \delta\psi^2$.

$$x = \left(\frac{3\cos\gamma}{2\,\delta\,\sigma\,\lambda^{1/2}}\right)^{1/2} \int_{\gamma}^{2\tan^{-1}\sqrt{[(\psi_m-\psi)\cos\gamma/\sigma]}} \frac{\mathrm{d}T}{\sqrt{(1-\epsilon^2\sin^2 T)}}$$

or

$$\psi_m - \psi = \sigma\,(\tan^2 T/2)/\cos\gamma$$

$$T = \sin^{-1}\left[\operatorname{sn} x\left(\frac{2\,\sigma\,\delta\,\gamma^{1/2}}{3\cos\gamma}\right)^{1/2}\right]$$

where

$$\sigma^2 = \tfrac{3}{4}\,[\psi_m^2 + \psi_m/\delta + 4/\delta - 3/(4\delta^2)]$$

$$\tan\gamma = 3\,[\psi_m + 1/(2\delta)]/2\sigma$$

$$\epsilon^2 = \tfrac{1}{2}\,(1 + \sin\gamma).$$

The current parameter λ and the unknown maximum temperature is obtained by putting $x = 1$ and $\psi = 0$. The graph of the equation so obtained is compared for ($\delta = 0{\cdot}195$) with inequalities (11) in Fig. 1. We note that solutions do not exist when the current parameter exceeds 1·28. For each value of λ below the critical there are just two solutions. A double-valued solution for currents below the critical is common to all known exact solutions.

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Résumé—L'effet de la dépendance non-linéaire de la résistance en fonction de la température sur la production de chaleur par effet Joule dans des systèmes conducteurs de l'électricité est étudié. La théorie est comparée avec des théories linéaires bien connues. Dans les matériaux conducteurs habituels, il existe un courant critique au-delà duquel on ne peut pas obtenir de solutions permanentes. A la différence de la théorie linéaire, le courant critique n'implique pas d'avoir des températures illimitées. La théorie linéaire surrestime toujours le courant critique. Dans la théorie non-linéaire, les solutions pour des courants au-dessous de la valeur critique ne sont pas uniques. Les différentes branches de ces solutions ne sont pas toutes stables. Il existe une solution neutre et des solutions au voisinage de l'instabilité pour le problème associé de la stabilité lorsque le courant est à la valeur critique.

Zusammenfassung—Der Einfluss einer nicht-linearen Abhängigkeit des Widerstandes von der Temperatur auf die Erzeugung Joule'scher Wärme in einem elektrisch leitenden System wird untersucht. Diese Theorie wird mit bekannten linearen Theorien verglichen.

Bei gewöhnlichen leitenden Stoffen gibt es einen kritischen Strom, jenseits dessen stetige Lösungen nicht mehr erreichbar sind. Im Gegensatz zur linearen Theorie bedeutet kritischer Strom nicht unbegrenzte Temperaturen. Die lineare Theorie überschätzt den kritischen Strom immer. Bei der nicht-linearen Theorie sind die Lösungen für Ströme unterhalb des kritischen nicht eindeutig. Die verschiedenen Äste der nicht eindeutigen Lösungen sind nicht alle stabil. Wenn der Strom den kritischen Wert erreicht hat, gibt es für das zusammengefasste Stabilitätsproblem eine neutrale und benachbarte instabile Lösung.

Аннотация—Исследуется влияние нелинейной зависимости сопротивления от температуры на Джоулево выделение тепла в электропроводящих системах. Теоретические результаты данного исследования сравниваются с результатами хорошо известных линейных теорий. В обычных проводящих материалах существует критический ток, для значений ниже которого стационарных решений нет. В противоположность линейной теории в данном случае критический ток не подразумевает неограниченных температур. Линейная теория всегда завышает критический ток. По нелинейной теории решения для токов ниже критического не однозначны. Не все ветви неоднозначных решений стабильны. Существуют нейтральное и смежное нестабильные решения в соответствующеи задаче устойчивости при критических значениях тока.